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On spherical harmonic representation of transient waves in dispersive media

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Abstract
Axisymmetric transient solutions to the inhomogeneous telegraph equation are constructed in terms of spherical harmonics. Explicit solutions of the initial-value problem are derived in the spacetime domain by means of the Smirnov method of incomplete separation of variables and the Riemann formula. The corresponding Riemann function is constructed with the help of the Olevsky theorem. Solutions for some source distributions on a sphere expanding with a velocity greater than the wavefront velocity are obtained. This allows an analogous solution in the case of a circle belonging to a sphere expanding with the wavefront velocity to be written at once. Application of the scalar solution to a description of electromagnetic waves is also discussed.

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1. Introduction

In this paper we derive axisymmetric transient solutions to the inhomogeneous telegraph equation in terms of spherical harmonics. It extends the results obtained earlier in [1] for the inhomogeneous wave equation to the case of dispersive media. Explicit solutions of the initial-value problem are constructed in the spacetime domain by application of the Smirnov method of incomplete separation of variables [2]. In the case in question the problem is formulated in spherical coordinates and the polar angle is separated by the expansion in terms of Legendre's polynomials. The resulting problem for the expansion coefficients is solved with the help of the Riemann formula [3, 4]; the Riemann function is constructed using the Olevsky theorem [5]. Solutions for some source distributions on a sphere expanding with a velocity greater than the wavefront velocity are obtained. Hence we get description of waves produced by sources on an expanding circle formed as an intersection of a conical surface with the sphere. This allows an analogous solution in the case of a circle belonging to a sphere expanding with the wavefront velocity to be written immediately. We discuss the application of the scalar solution to the description of electromagnetic waves. Previously obtained analogous partial solutions were constructed for sources on a sphere expanding with a velocity less than the wavefront velocity [6].

2. Basic relations

After elimination of a possible term proportional to $\partial_\tau \psi$ by means of suitable transformation, we write the 3D telegraph equation in the form

$$(\partial_\tau^2 - a^2 - \triangle)\psi = (4\pi/c)j. \tag{1}$$

Here

$$\triangle = (1/r^2)\partial_r(r^2\partial_r) + (1/r^2\sin\theta)\partial_\theta(\sin\theta\,\partial_\theta) \tag{2}$$

is the Laplace operator, the constant a^2 determines the dispersion, r and θ are the radial variable and the polar angle in the spherical coordinates, $\tau = ct$ is the time variable, and c is the wavefront velocity (for electromagnetic waves, the velocity of light). The initial condition is

$$\psi = j \equiv 0 \qquad \tau < 0. \tag{3}$$

Since the interval of the radial variable is $r > 0$, one needs a boundary condition. We suppose that

$$\psi \longrightarrow 0 \quad r \longrightarrow 0+\,. \tag{4}$$

Representing the wavefunction ψ and the source function j in terms of Legendre's polynomials $P_n(\cos\theta)$

$$\psi = \sum_{n=0}^{\infty} P_n(\cos\theta)\psi_n(r,\tau) \tag{5}$$

$$j = \sum_{n=0}^{\infty} P_n(\cos\theta) j_n(r,\tau) \tag{6}$$

and writing the expansion coefficients of expression (5) as $\psi_n = u_n/r$, we separate the angle variable in (1), (3) and (4) and obtain the problem for the functions $u_n(r,\tau)$

$$\left(\partial_\tau^2 - \partial_r^2 + n(n+1)/r^2 - a^2\right)u_n = (4\pi/c)rj_n \qquad u_n = j_n \equiv 0 \quad \tau < 0 \tag{7}$$

$$u_n \to 0 \quad r \to 0+\,. \tag{8}$$

We have to satisfy boundary condition (8) by choosing the function $rj_n(r,\tau)$ on the interval $r < 0$. The latter does not correspond to real space.

One can write a solution of the above problem with the help of the Riemann formula [3, 4]

$$u_n = \frac{2\pi}{c}\int\!\!\int_\triangle \mathrm{d}\tau'\,\mathrm{d}r' G_n(r,\tau,r',\tau'). \tag{9}$$

Here the integrand denotes

$$G_n = r' j_n(r',\tau') R_n(r,\tau,r',\tau') \tag{10}$$

$$R_n = P_n\left(\frac{r^2 + r'^2 - (\tau-\tau')^2}{2rr'}\right) + \int_{\tau-\tau'}^{r-r'} \mathrm{d}\xi\, P_n\left(\frac{r^2+r'^2-\xi^2}{2rr'}\right)\frac{\partial}{\partial\xi} I_0\!\left(a\sqrt{(\tau-\tau')^2-\xi^2}\right) \tag{11}$$

where $I_0(x)$ is the modified Bessel function of the first kind and $P_n(x)$ the Legendre function. The term R_n denotes the Riemann function [4, 7] that for partial differential equations of the hyperbolic type can be treated as an extension of the Green function (see [3], item 9.3-3, and [8]) that takes into account initial conditions given on the boundary curve (remarkably, in some textbooks and manuals the Riemann function is referred to as the Riemann–Green function, see, e.g., [3], item 10.3-6). In most cases of application of this technique, derivation of the Riemann function comprises the major difficulty in constructing the solution. However, in the particular case of the telegraph equation, the Olevsky theorem [5] allows us to build R_n on the basis of known Riemann functions for simpler differential operators $\partial_\tau^2 - \partial_r^2 + n(n+1)/r^2$ and $\partial_\tau^2 - \partial_r^2 - a^2$. The triangular integration domain on the r', τ'-plane is confined by the characteristics $\tau' \mp r' = \tau \mp r$ and the axis $\tau' = 0$.

These expressions are the basic relations for constructing waves formed by different sources.

3. Representation of coefficients Ψ_n for sources on an expanding sphere

Let us suppose that the source is distributed on a sphere expanding with a constant velocity $v = c\beta$, so that $r = r_0 + \beta\tau$. Here $r_0 > 0$ is a constant and $\beta \in (0, \infty)$ is the dimensionless velocity parameter. Writing the source of equation (1) as

$$j = (1/2\pi r^2)F(r, \theta, \tau)\delta(r - r_0 - \beta\tau) \tag{12}$$

where $\delta(x)$ is the Dirac function, we obtain the source function of equation (7)

$$rj_n = (1/2\pi(r_0 + \beta\tau))F_n(r_0 + \beta\tau, \tau)\delta(r - r_0 - \beta\tau). \tag{13}$$

The factor j_n is defined from series (6). Note that boundary condition (8) may be incorrect at the initial moment of time if $r_0 = 0$.

Taking into account expression (13) for $r' > 0$, we write the integrand of expression (9) as

$$\tilde{G}_n^{(+)}(r, \tau, r', \tau') = G_n^{(+)}(\tau')\delta(r' - r_0 - \beta\tau') \tag{14}$$

while for $r' < 0$ we suppose

$$\tilde{G}_n^{(-)}(r, \tau, r', \tau') = A_n G_n^{(-)}(\tau')\delta(r' + r_0 + \beta\tau') \tag{15}$$

A_n is a constant,

$$G_n^{(\pm)}(\tau') = \frac{1}{r_0 + \beta\tau'} F_n(r_0 + \beta\tau', \tau') R_n^{(\pm)}(r, \tau, r_0 + \beta\tau', \tau')$$

and $R_n^{(\pm)}$ denotes

$$R_n^{(\pm)} = P_n\left(\pm\frac{r^2 + (r_0 + \beta\tau')^2 - (\tau - \tau')^2}{2r(r_0 + \beta\tau')}\right) + \int_{\tau-\tau'}^{r\mp(r_0+\beta\tau')} \mathrm{d}\xi\, P_n\left(\pm\frac{r^2 + (r_0 + \beta\tau')^2 - \xi^2}{2r(r_0 + \beta\tau')}\right)\frac{\partial}{\partial\xi} I_0\Big(a\sqrt{(\tau - \tau')^2 - \xi^2}\Big). \tag{16}$$

Using expressions (9), (14) and (15) we write coefficients of series (5) in the general form

$$\psi_n = \frac{1}{cr}\iint_\Delta \mathrm{d}\tau'\,\mathrm{d}r'\tilde{G}_n^{(+)}(r, \tau, r', \tau') + \frac{1}{cr}\iint_\Delta \mathrm{d}\tau'\,\mathrm{d}r'\tilde{G}_n^{(-)}(r, \tau, r', \tau'). \tag{17}$$

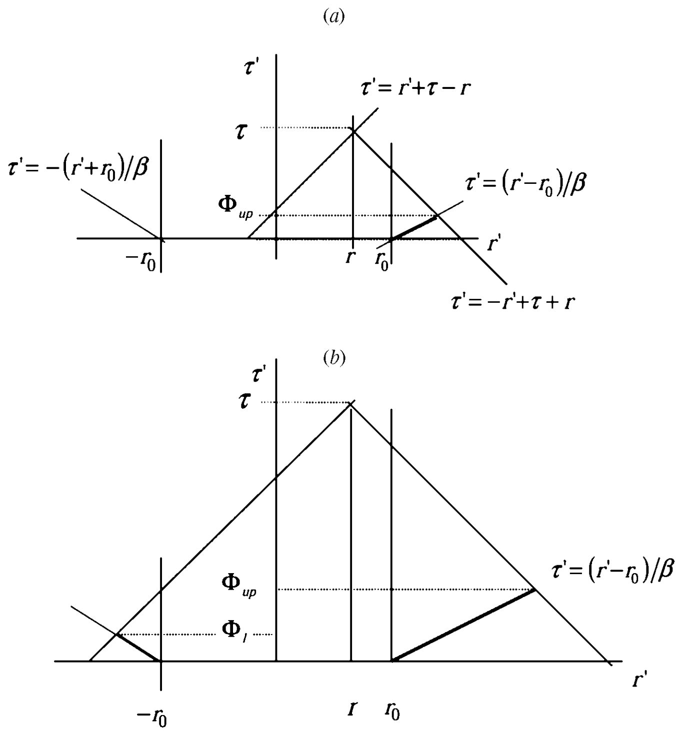

Figure 1. The r', r'-plane diagrams for $r < r_0$, (*a*) $r_0 + r > \tau > r_0 - r$, (*b*) $\tau > r_0 + r$; $\psi \equiv 0$, $\tau < r_0 - r$.

4. Discussion on applications

Here we use the results obtained for the description of explicit solutions to the telegraph equation.

4.1. Sources on a sphere expanding with a velocity greater than the wavefront velocity

In this case $\beta > 1$. Depending on the interrelation between the space variable r and the constant r_0, we distinguish two main cases:

(1) The observation point lies inside the sphere $r = r_0$, i.e. $r < r_0$. For the observation time $r + r_0 > \tau > r_0 - r$ the integration domain on the r', τ'-plane is shown in figure 1(*a*). Here we use the first item of expression (17) only. Performing integration with respect to the space variable we obtain

$$\psi_n = \psi_n^+ = \frac{1}{cr} \int_{\Phi_l}^{\Phi_{\rm up}} {\rm d}\tau' G_n^{(+)}(\tau') \tag{18}$$

where $\Phi_{\rm up} = (\tau + r - r_0)/(\beta + 1)$ and $\Phi_l = 0$.

The upper limit of integration is the time coordinate of the intersection point of the straight line $\tau' = (r' - r_0)/\beta$ and the characteristic $\tau' + r' = \tau + r$.

For the case $\tau > r_0 + r$ the domain of integration is shown in figure 1(*b*). Here we use both the first and the second terms of expression (17) and write

$$\psi_n = \psi_n^+ + \frac{1}{cr} A_n \int_0^{(\tau - r - r_0)/(\beta+1)} \mathrm{d}\tau' G_n^{(-)}(\tau') \tag{19}$$

where the upper limit of the second item is the coordinate of the intersection of the line $\tau' = -(r' - r_0)/\beta$ and the characteristic $\tau' - r' = \tau - r$. It may be verified that we satisfy the boundary condition (8) by choosing $A_n = (-1)^{n+1}$. This also allows the transformation of expression (19) (see the appendix).

(2) The integration domains for the opposite case $r > r_0$ are shown in figures 2(*a*)–(*c*). It is clear that we have to use the coefficient $A_n = (-1)^{n+1}$ as well. When the intervals of observation time are $r - r_0 > \tau > (r - r_0)/\beta$ and $r + r_0 > \tau > r - r_0$, we use the first item of expression (17). It follows from figures 2(*a*) and (*b*) that

$$\psi_n = \frac{1}{cr} \int_{\Phi_l}^{(\tau + r - r_0)/(\beta+1)} \mathrm{d}\tau' G_n^{(+)}(\tau') \tag{20}$$

where

$$\Phi_l = \begin{cases} (r - r_0 - \tau)/(\beta - 1) & r - r_0 > \tau > (r - r_0)/\beta \\ 0 & r + r_0 > \tau > r - r_0. \end{cases} \tag{21}$$

Finally, when $\tau > r + r_0$ we get ψ_n in the form (19), that one can see by comparing figures 1(*b*) and 2(*c*).

Collecting expressions (18)–(21) and expansion (5) we obtain the solution of the inhomogeneous telegraph equation in terms of the transient spherical harmonics for source distributions on an expanding sphere.

4.2. *Sources on a circle, $\beta > 1$*

On the basis of the constructed solution it is easy to get expressions describing waves formed by source distributions on an expanding circle which is the intersection of the sphere $\tau = r_0 + \beta\tau$ and the conical surface $\theta = \theta_0$, $\theta_0 \in (0, \pi)$. Writing the source function of equation (1) as

$$j = (1/(2\pi r^2)) f(r, \tau) \delta(\cos\theta - \cos\theta_0) \delta(r - r_0 - \beta\tau) \tag{22}$$

one gets

$$r j_n = (1/(2\pi r))(n + 1/2) P_n(\cos\theta_0) f(r, \tau) \delta(r - r_0 - \beta\tau). \tag{23}$$

This allows the solution of the telegraph equation to be written immediately by means of the coefficients ψ_n obtained in the previous subsection substituting the integrands

$$G_n^{(\pm)} = (n + 1/2) P_n(\cos\theta_0) \frac{1}{r_0 + \beta\tau'} f(r_0 + \beta\tau', \tau') R_n^{(\pm)}(r, \tau, r_0 + \beta\tau', \tau'). \tag{24}$$

4.3. *On solutions of analogous problems for $\beta = 1$*

Applying the results obtained in the previous subsections, one can get solutions for the sphere expanding with the wavefront velocity as well as for a circle belonging to the sphere. Here one has to omit the case $r > r_0$, $r - r_0 > \tau > (r - r_0)/\beta$ shown in figure 2(*a*), and suppose

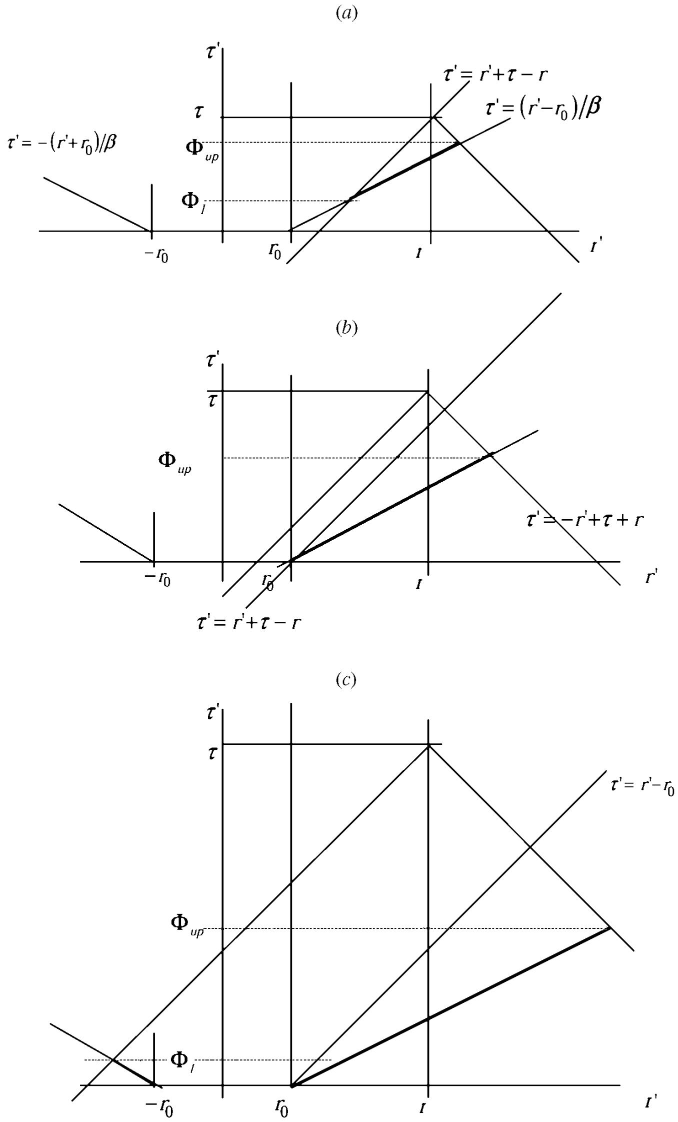

Figure 2. The r', r'-plane diagrams for $r > r_0$, (*a*) $r - r_0 > \tau > (r - r_0)/\beta$, (*b*) $r + r_0 > \tau > r - r_0$, (*c*) $\tau > r + r_0$; $\psi \equiv 0$, $\tau < (r - r_0)/\beta$.

$\beta = 1$. So, with expressions (18) and (19) we obtain the solution in the form (5) using the integrands

$$G_{nc}^{(\pm)} = \frac{1}{r_0 + r'} f(r_0 + r', \tau') P_n \left(\pm \frac{r^2 + r_0^2 - \tau^2 + 2(r_0 + \tau)\tau'}{2r(r_0 + \tau')} \right) + \int_{\tau - \tau'}^{r - r_0 - \tau'} \mathrm{d}\xi \, P_n \left(\pm \frac{r^2 + (r_0 + \tau')^2 - \xi^2}{2r(r_0 + \tau')} \right) \frac{\partial}{\partial \xi} I_0\big(a\sqrt{(\tau - \tau')^2 - \xi^2}\big). \tag{25}$$

The upper integration limit of the integrals Ψ_n for both $r_0 > r$ and $r_0 < r$ is

$$\Phi_{\mathrm{up}} = (\tau + r - r_0)/2. \tag{26}$$

The lower limits are

$$\Phi_l = 0 \tag{27}$$

in the case of $r_0 > r$ and $r_0 + r > \tau > r_0 - r$ as well as $r_0 < r$ and $r_0 + r > \tau > r - r_0$, and

$$\Phi_l = (\tau - r - r_0)/2 \tag{28}$$

as for both $r_0 > r$ and $r_0 < r$ provided that $\tau > r + r_0$.

Note that we get the analogous solutions of the wave equation obtained in [9], by supposing the constant $a = 0$ in the expressions of the above subsections.

5. Application to electromagnetic waves

As an example we discuss the construction of the axisymmetric solution of Maxwell's equations for a conducting medium

$$\begin{aligned}&(1/(r\sin\theta))\partial_\theta(\sin\theta B_\varphi) = \partial_\tau E_r + (4\pi\sigma/c)E_r + (4\pi/c)j_r\\&-(1/r)\partial_r(rB_\varphi) = \partial_\tau E_\theta + (4\pi\sigma/c)E_\theta + (4\pi/c)j_\theta\\&(1/r)(\partial_r(rE_\theta) - \partial_\theta E_r) = -\partial_\tau B_\varphi.\end{aligned} \tag{29}$$

Here E_r, E_θ and B_φ are the components of the electric field strength and the magnetic induction vectors, $\mathbf{j} = \mathbf{e}_r j_r + \mathbf{e}_\theta j_\theta$ is the current density vector, $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are the unit vectors, and σ is the conductivity.

Supposing $j_\theta = 0$, we obtain from equations (29)

$$\left(\partial_r^2 + (1/r^2)\partial_\theta((1/\sin\theta)\partial_\theta\sin\theta) - \partial_\tau^2 - (4\pi\sigma/c)\partial_\tau\right)rB_\varphi = (4\pi/c)\partial_\theta j_r.$$

This equation and the expansions

$$rB_\varphi = \exp(-2\pi\sigma\tau/c)\sum_{n=1}^{\infty} w_n(r,\tau)P_n^1(\cos\theta)$$

and

$$j_r = \sum_{n=0}^{\infty} j_n(r,\tau)P_n(\cos\theta)$$

where $P_n^1(\cos\theta) = \partial_\theta P_n(\cos\theta)$, $j_n = (n+1/2)\int_{-1}^{+1} \mathrm{d}x\, P_n(x) j_r(r,x,\tau)$ and $x = \cos\theta$, together with the boundary and initial conditions yield the problem for the coefficients $w_n(r,\tau)$

$$\begin{aligned}&\left(\partial_\tau^2 - \partial_r^2 + n(n+1)/r^2 - (2\pi\sigma/c)\right)w(r,\tau) = -(4\pi/c)j_n(r,\tau)\exp(2\pi\sigma\tau/c)\\&w_n(r,\tau) = j_n(r,\tau) = 0 \quad \tau < 0 \qquad w_n(r,\tau) \to 0 \quad r \to 0+.\end{aligned} \tag{30}$$

This problem is analogous to (7) and (8), that permit us to obtain B_φ by using the scalar solutions. One can, in principal, construct E_θ and E_r from the second and the third equations of system (29).

Note that electromagnetic wave sources distributed on a circle (or localized at some point of the circle) belonging to a fictitious sphere expanding with a velocity greater than the velocity of light can be realized with the help of a pulse of hard radiation (gamma rays) with the conical front directed to the absorption domains of various geometries (a cone or part thereof, some helical lines, or a cylindrical surface). The ionization fronts producing photoelectrons are defined by intersection of the front of the gamma-ray pulse with the absorption domain and, like shadow fronts, are free from restrictions imposed by the theory of relativity on the velocity of energy or information transport. The photoelectrons that follow the ionization front have a non-isotropic velocity distribution with a maximum in the direction of the photon propagation, which results in the appearance of a macroscopic source current. Such models of formation of transient currents launching electromagnetic waves were first developed for calculation of electromagnetic radiation accompanying nuclear explosions [10, 11]. A more profound investigation, including all main processes of interaction of 10 MeV photons with matter (photoabsorption, Compton scattering and pair production), was recently made by Valiev [12].

6. Concluding remarks

In the above sections we have considered only axisymmetric solutions. Notably, the results of the scalar problem can be extended to the more general non-axisymmetric case, in which the Laplace operator of the telegraph equation (1) is

$$\triangle = (1/r^2)\partial_r(r^2\partial_r) + (1/(r^2\sin\theta))\partial_\theta(\sin\theta\,\partial_\theta) + (1/(r^2\sin\theta))\partial^2_\varphi$$

and we represent the solution ψ and the source function j in terms of the associated Legendre functions of the first kind $P_n^m(\cos\theta)$ and $\mathrm{e}^{\mathrm{i}m\varphi}$ by

$$\begin{aligned} \psi &= \sum_{n,m} P_n^m(\cos\theta)\,\mathrm{e}^{\mathrm{i}m\varphi}\psi_{nm}(r,\tau) \\ j &= \sum_{n,m} P_n^m(\cos\theta)\,\mathrm{e}^{\mathrm{i}m\varphi} j_{nm}(r,\tau) \end{aligned} \tag{31}$$

where n and m are integers such that $n \geqslant m \geqslant 0$, and φ is the azimuth angle. Writing $\psi_{nm} = u_{nm}/r$ and using the initial and boundary conditions, we obtain the problem for the function u_{nm}

$$\begin{aligned} &\left(\partial^2_\tau - \partial^2_r + n(n+1)/r^2 - a^2\right)u_{nm} = (4\pi/c)r j_{nm} \\ &u_{nm} = j_{nm} \equiv 0 \quad \tau < 0 \qquad u_{nm} \to 0 \quad r \to 0+ \end{aligned}$$

in which the differential operator of the equation does not depend on m. The solution of the above problem is

$$u_{nm} = \frac{2\pi}{c}\iint_\triangle \mathrm{d}\tau'\,\mathrm{d}r'\,G_{nm}(r,\tau,r',\tau') \tag{32}$$

where the integrand $G_{nm} = r' j_{nm}(r',\tau')R_n(r,\tau,r',\tau')$ depends on m only through the expansion coefficient j_{nm}.

Collecting expressions (31) and (32) and the results of sections 3 and 4, one can obtain the non-axisymmetric solutions of the telegraph equation in terms of the spherical harmonics. Application of these solutions to the electromagnetic waves for an arbitrary current density vector is a problem that requires individual investigation.

Appendix

Let us verify that the boundary condition $\psi_n \to 0\ r- > 0+$ is satisfied when the coefficient $A_n = (-1)^{n+1}$. We write $\psi_n = \psi_{\mathrm{on}} + \psi_{\mathrm{an}}$, where the second term determines the wave dispersion while the first term gives the solution of the wave equation. The latter was investigated in [13] (see also [1]), wherefore we discuss ψ_{an} only. It is clear from figures 1 and 2 that we have to discuss the case $r < r_0$ for the time interval $\tau > r + r_0$ (see figure 1(*b*)). Hence using expressions (18) and (19) we write $\psi_{\alpha n}$ in the form

$$\psi_{\mathrm{an}} = \frac{1}{cr}\int_0^{(\tau+r-r_0)/(\beta+1)} \mathrm{d}\tau'\,G^{(+)}_{\mathrm{an}}(\tau') + \frac{1}{cr}A_n\int_0^{(\tau-r-r_0)/(\beta+1)} \mathrm{d}\tau'\,G^{(-)}_{\mathrm{an}}(\tau') \tag{33}$$

where

$$G_{\rm an}^{(\pm)} = \frac{1}{r_0 + \beta\tau'} F_n(r_0 + \beta\tau', \tau') \times \int_{\tau-\tau'}^{(\tau\mp(r_0+\beta\tau'))} \mathrm{d}\xi \, P_n\left(\pm\frac{r^2 + (r_0 + \beta\tau')^2 - \xi^2}{2r(r_0 + \beta\tau')}\right) \frac{\partial}{\partial\xi} I_0\big(a\sqrt{(\tau - \tau')^2 - \xi^2}\big).$$

Remembering that $P_n(-x) = (-1)^n P_n(x)$ and performing derivation with respect to the variable ξ, one can transform $\psi_{\rm an}$ into

$$\psi_{\rm an} = \frac{a}{c} \int_0^{(\tau+r-r_0)/(\beta+1)} \mathrm{d}\tau' F_n(r_0 + \beta\tau', \tau') \int_{x_l}^{1} \mathrm{d}x \, Q_{\rm an}(\tau', x) - \frac{a}{c} \int_0^{(\tau-r-r_0)/(\beta+1)} \mathrm{d}\tau' F_n(r_0 + \beta\tau', \tau') \int_{x_l}^{-1} \mathrm{d}x \, Q_{\rm an}(\tau', x). \tag{34}$$

Here

$$Q_{\rm an}(\tau', x) = P_n(x) \frac{I_1\big(a\sqrt{(\tau - \tau')^2 - r^2 - (r_0 + \beta\tau')^2 + 2r(r_0 + \beta\tau')x}\big)}{\sqrt{(\tau - \tau')^2 - r^2 - (r_0 + \beta\tau')^2 + 2r(r_0 + \beta\tau')x}} \tag{35}$$

the integration variable is

$$x = \frac{1}{2r(r_0 + \beta\tau')}(r^2 + (r_0 + \beta\tau')^2 - \xi^2)$$

and the lower limit of the internal integral is

$$x_l = \frac{1}{2r(r_0 + \beta\tau')}(r^2 + (r_0 + \beta\tau')^2 - (\tau - \tau')^2).$$

Hence one can see that $x_l = \pm 1$ if $\tau' = (\tau \pm r - r_0)/(\beta + 1)$ and $x_l < -1$ if $\tau' = 0$ provided that $\tau > r + r_0$. Since the observation time $\tau > r + r_0$ we rewrite expression (34) as

$$\psi_{\rm an} = \frac{a}{c} \int_{(\tau-r-r_0)/(\beta+1)}^{(\tau+r-r_0)/(\beta+1)} \mathrm{d}\tau' F_n(r_0 + \beta\tau', \tau') \int_{-1}^{1} \mathrm{d}x \, Q_{\rm an}(\tau', x) + \frac{a}{c} \int_0^{(\tau-r-r_0)/(\beta+1)} \mathrm{d}\tau' F_n(r_0 + \beta\tau', \tau') \int_{x_l}^{-1} \mathrm{d}x \, Q_{\rm an}(\tau', x) - \frac{a}{c} \int_0^{(\tau-r-r_0)/(\beta+1)} \mathrm{d}\tau' F_n(r_0 + \beta\tau', \tau') \int_{x_l}^{-1} \mathrm{d}x \, Q_{\rm an}(\tau', x)$$

and putting (35) obtain

$$\psi_{\rm an} = \frac{a}{c} \int_{(\tau-r-r_0)/(\beta+1)}^{(\tau+r-r_0)/(\beta+1)} \mathrm{d}\tau' F_n(r_0 + \beta\tau', \tau') \int_{-1}^{1} \mathrm{d}x \, P_n(x) \times \frac{I_1\big(a\sqrt{(\tau - \tau')^2 - r^2 - (r_0 + \beta\tau')^2 + 2r(r_0 + \beta\tau')x}\big)}{\sqrt{(\tau - \tau')^2 - r^2 - (r_0 + \beta\tau')^2 + 2r(r_0 + \beta\tau')x}}$$

where $x \in (-1, 1)$. When r tends to zero, this expression tends to zero provided that F_n is a finite function.

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